CHAPTER 5

Series Solutions of ODEs. Special Functions

In the previous chapters, we have seen that linear ODEs with *constant coefficients* can be solved by algebraic methods, and that their solutions are elementary functions known from calculus. For ODEs with *variable coefficients* the situation is more complicated, and their solutions may be nonelementary functions. *Legendre's, Bessel's*, and the *hypergeometric equations* are important ODEs of this kind. Since these ODEs and their solutions, the *Legendre polynomials, Bessel functions*, and *hypergeometric functions*, play an important role in engineering modeling, we shall consider the two standard methods for solving such ODEs.

The first method is called the **power series method** because it gives solutions in the form of a power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$.

The second method is called the **Frobenius method** and generalizes the first; it gives solutions in power series, multiplied by a logarithmic term $\ln x$ or a fractional power x^r, in cases such as Bessel's equation, in which the first method is not general enough.

All those more advanced solutions and various other functions not appearing in calculus are known as *higher functions* or **special functions**, which has become a technical term. Each of these functions is important enough to give it a name and investigate its properties and relations to other functions in great detail (take a look into Refs. [GenRef1], [GenRef10], or [All] in App. 1). Your CAS knows practically all functions you will ever need in industry or research labs, but it is up to you to find your way through this vast terrain of formulas. The present chapter may give you some help in this task.

COMMENT. You can study this chapter directly after Chap. 2 because it needs no material from Chaps. 3 or 4.

Prerequisite: Chap. 2.
Section that may be omitted in a shorter course: 5.5.
References and Answers to Problems: App. 1 Part A, and App. 2.

5.1 Power Series Method

power series可用來解微分方程

The **power series method** is the standard method for solving linear ODEs with ***variable*** coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we shall see. In this section we begin by explaining the idea of the power series method.

From calculus we remember that a **power series** (in powers of $x - x_0$) is an infinite series of the form

$$\boxed{\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.} \tag{1}$$

Here, x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **center** of the series. In particular, if $x_0 = 0$, we obtain a **power series in powers of** $\boldsymbol{x}$

通常用(2)來解微分方程。

$$\boxed{\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.} \tag{2}$$

We shall assume that all variables and constants are real.

We note that the term "power series" usually refers to a series of the form (1) [or (2)] but ***does not include*** series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the Legendre and Bessel equations for integer values of the parameter.

EXAMPLE 1 **Familiar Power Series** are the Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots.$$

Idea and Technique of the Power Series Method

The idea of the power series method for solving linear ODEs seems natural, once we know that the most important ODEs in applied mathematics have solutions of this form. We explain the idea by an ODE that can readily be solved otherwise.

EXAMPLE 2 **Power Series Solution.** Solve $y' - y = 0$.

Solution. In the first step we insert

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m \tag{2}$$

and the series obtained by termwise differentiation

(3)
$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{m=1}^{\infty} ma_m x^{m-1}$$

into the ODE:

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) - (a_0 + a_1x + a_2x^2 + \cdots) = 0.$$

原方程式的解是exp(x)
用 power series代入求解。
發現，答案為 exp(x)的泰勒展開式。

Then we collect like powers of x, finding

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0.$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0, \cdots.$$

Solving these equations, we may express $a_1, a_2, \cdots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots.$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) = a_0e^x.$$

Test your comprehension by solving $y'' + y = 0$ by power series. You should get the result $y = a_0 \cos x + a_1 \sin x$.

We now describe the method in general and justify it after the next example. For a given ODE

(4)
$$y'' + p(x)y' + q(x)y = 0$$

we first represent $p(x)$ and $q(x)$ by power series in powers of x (or of $x - x_0$ if solutions in powers of $x - x_0$ are wanted). Often $p(x)$ and $q(x)$ are polynomials, and then nothing needs to be done in this first step. Next we assume a solution in the form of a power series (2) with unknown coefficients and insert it as well as (3) and

(5)
$$y'' = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

into the ODE. Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero, starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of (3) successively.

EXAMPLE 3 **A Special Legendre Equation.** The ODE

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

這一題下一小節會教。

occurs in models exhibiting spherical symmetry. Solve it.

Solution. Substitute (2), (3), and (5) into the ODE. $(1 - x^2)y''$ gives two series, one for y'' and one for $-x^2y''$. In the term $-2xy'$ use (3) and in $2y$ use (2). Write like powers of x vertically aligned. This gives

$$\begin{aligned} y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots \\ -x^2y'' &= \qquad\qquad\quad - 2a_2x^2 - 6a_3x^3 - 12a_4x^4 - \cdots \\ -2xy' &= \qquad - 2a_1x - 4a_2x^2 - 6a_3x^3 - 8a_4x^4 - \cdots \\ 2y &= 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots. \end{aligned}$$

Add terms of like powers of x. For each power $x^0, x, x^2, \cdots$ equate the sum obtained to zero. Denote these sums by [0] (constant terms), [1] (first power of x), and so on:

Sum	Power	Equations	
[0]	$[x^0]$	$a_2 = -a_0$	
[1]	$[x]$	$a_3 = 0$	
[2]	$[x^2]$	$12a_4 = 4a_2,$	$a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
[3]	$[x^3]$	$a_5 = 0$	since $a_3 = 0$
[4]	$[x^4]$	$30a_6 = 18a_4,$	$a_6 = \frac{18}{30}a_4 = \frac{18}{30}(-\frac{1}{3})a_0 = -\frac{1}{5}a_0.$

This gives the solution

$$y = a_1x + a_0(1 - x^2 - \tfrac{1}{3}x^4 - \tfrac{1}{5}x^6 - \cdots).$$

a_0 and a_1 remain arbitrary. Hence, this is a general solution that consists of two solutions: x and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$. These two solutions are members of families of functions called *Legendre polynomials* $P_n(x)$ and *Legendre functions* $Q_n(x)$; here we have $x = P_1(x)$ and $1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots = -Q_1(x)$. The minus is by convention. The index 1 is called the *order* of these two functions and here the order is 1. More on Legendre polynomials in the next section. ■

Theory of the Power Series Method

以下原理複變都教過。

The ***n*th partial sum** of (1) is

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \tag{6}$$

where $n = 0, 1, \cdots$. If we omit the terms of s_n from (1), the remaining expression is

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots. \tag{7}$$

This expression is called the **remainder** *of* (1) *after the term* $a_n(x - x_0)^n$.

For example, in the case of the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

we have

$$\begin{aligned} s_0 &= 1, & R_0 &= x + x^2 + x^3 + \cdots, \\ s_1 &= 1 + x, & R_1 &= x^2 + x^3 + x^4 + \cdots, \\ s_2 &= 1 + x + x^2, & R_2 &= x^3 + x^4 + x^5 + \cdots, \qquad \text{etc.} \end{aligned}$$

In this way we have now associated with (1) the sequence of the partial sums $s_0(x), s_1(x), s_2(x), \cdots$. If for some $x = x_1$ this sequence converges, say,

$$\lim_{n\to\infty} s_n(x_1) = s(x_1),$$

then the series (1) is called **convergent** *at* $x = x_1$, the number $s(x_1)$ is called the **value** or *sum* of (1) at x_1, and we write

$$s(x_1) = \sum_{m=0}^{\infty} a_m(x_1 - x_0)^m.$$

Then we have for every n,

$$s(x_1) = s_n(x_1) + R_n(x_1). \tag{8}$$

If that sequence diverges at $x = x_1$, the series (1) is called **divergent** at $x = x_1$.

In the case of convergence, for any positive ϵ there is an N (depending on ϵ) such that, by (8)

$$|R_n(x_1)| = |s(x_1) - s_n(x_1)| < \epsilon \qquad \text{for all } n > N. \tag{9}$$

Geometrically, this means that all $s_n(x_1)$ with $n > N$ lie between $s(x_1) - \epsilon$ and $s(x_1) + \epsilon$ (Fig. 104). Practically, this means that in the case of convergence we can approximate the sum $s(x_1)$ of (1) at x_1 by $s_n(x_1)$ as accurately as we please, by taking n large enough.

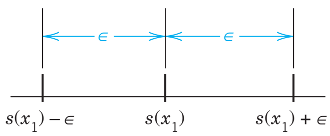

Fig. 104. Inequality (9)

Where does a power series converge? Now if we choose $x = x_0$ in (1), the series reduces to the single term a_0 because the other terms are zero. Hence the series converges at x_0. In some cases this may be the only value of x for which (1) converges. If there are other values of x for which the series converges, these values form an interval, the **convergence interval**. This interval may be finite, as in Fig. 105, with midpoint x_0. Then the series (1) converges for all x in the interior of the interval, that is, for all x for which

$$|x - x_0| < R \tag{10}$$

and diverges for $|x - x_0| > R$. The interval may also be infinite, that is, the series may converge for all x.

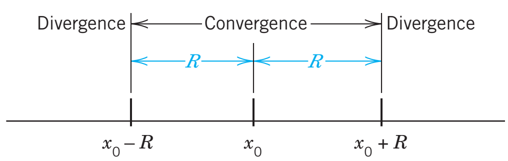

Fig. 105. Convergence interval (10) of a power series with center x_0

The quantity R in Fig. 105 is called the **radius of convergence** (because for a *complex* power series it is the radius of *disk* of convergence). If the series converges for all x, we set $R = \infty$ (and $1/R = 0$).

The radius of convergence can be determined from the coefficients of the series by means of each of the formulas

$$\text{(11)} \qquad \text{(a)} \quad R = 1\Big/ \lim_{m\to\infty} \sqrt[m]{|a_m|} \qquad \text{(b)} \quad R = 1\Big/ \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|$$

provided these limits exist and are not zero. [If these limits are infinite, then (1) converges only at the center x_0.]

EXAMPLE 4 Convergence Radius $R = \infty, 1, 0$

For all three series let $m \to \infty$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots, \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0, \qquad R = \infty$$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots, \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{1} = 1, \qquad R = 1$$

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + \cdots, \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+1)!}{m!} = m + 1 \to \infty, \qquad R = 0.$$

Convergence for all x ($R = \infty$) is the best possible case, convergence in some finite interval the usual, and convergence only at the center ($R = 0$) is useless. ■

When do power series solutions exist? *Answer:* if p, q, r in the ODEs

$$\text{(12)} \qquad y'' + p(x)y' + q(x)y = r(x)$$

have power series representations (Taylor series). More precisely, a function $f(x)$ is called **analytic** at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with positive radius of convergence. Using this concept, we can state the following basic theorem, in which the ODE (12) is **in standard form**, that is, it begins with the y''. If your ODE begins with, say, $h(x)y''$, divide it first by $h(x)$ and then apply the theorem to the resulting new ODE.

THEOREM 1

Existence of Power Series Solutions

If p, q, and r in (12) *are analytic at $x = x_0$, then every solution of* (12) *is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.*

The proof of this theorem requires advanced complex analysis and can be found in Ref. [A11] listed in App. 1.

We mention that the radius of convergence R in Theorem 1 is at least equal to the distance from the point $x = x_0$ to the point (or points) closest to x_0 at which one of the functions p, q, r, as functions of a *complex variable*, is not analytic. (Note that that point may not lie on the x-axis but somewhere in the complex plane.)

Further Theory: Operations on Power Series

In the power series method we differentiate, add, and multiply power series, and we obtain coefficient recursions (as, for instance, in Example 3) by equating the sum of the coefficients of each occurring power of x to zero. These four operations are permissible in the sense explained in what follows. Proofs can be found in Sec. 15.3.

1. Termwise Differentiation. *A power series may be differentiated term by term*. More precisely: if

$$y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m$$

converges for $|x - x_0| < R$, where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x:

$$y'(x) = \sum_{m=1}^{\infty} ma_m(x - x_0)^{m-1} \qquad (|x - x_0| < R).$$

Similarly for the second and further derivatives.

2. Termwise Addition. *Two power series may be added term by term*. More precisely: if the series

$$\sum_{m=0}^{\infty} a_m(x - x_0)^m \quad \text{and} \quad \sum_{m=0}^{\infty} b_m(x - x_0)^m \tag{13}$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then the series

$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges and represents $f(x) + g(x)$ for each x that lies in the interior of the convergence interval common to each of the two given series.

3. Termwise Multiplication. *Two power series may be multiplied term by term.* More precisely: Suppose that the series (13) have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$, that is,

$$a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 + \cdots$$

$$= \sum_{m=0}^{\infty} (a_0b_m + a_1b_{m-1} + \cdots + a_mb_0)(x - x_0)^m$$

converges and represents $f(x)g(x)$ for each x in the interior of the convergence interval of each of the two given series.

4. Vanishing of All Coefficients (*"Identity Theorem for Power Series."*) *If a power series has a positive radius of convergent convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.*

PROBLEM SET 5.1

1. WRITING AND LITERATURE PROJECT. Power Series in Calculus. (a) Write a review (2–3 pages) on power series in calculus. Use your own formulations and examples—do not just copy from textbooks. No proofs. **(b)** Collect and arrange Maclaurin series in a systematic list that you can use for your work.

2–5 REVIEW: RADIUS OF CONVERGENCE

Determine the radius of convergence. Show the details of your work.

2. $\sum_{m=0}^{\infty} (m+1)mx^m$

3. $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$

4. $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$

5. $\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$

6–9 SERIES SOLUTIONS BY HAND

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

6. $(1+x)y' = y$

7. $y' = -2xy$

8. $xy' - 3y = k$ (= const)

9. $y'' + y = 0$

10–14 SERIES SOLUTIONS

Find a power series solution in powers of x. Show the details.

10. $y'' - y' + xy = 0$

11. $y'' - y' + x^2y = 0$

12. $(1-x^2)y'' - 2xy' + 2y = 0$

13. $y'' + (1+x^2)y = 0$

14. $y'' - 4xy' + (4x^2-2)y = 0$

15. Shifting summation indices is often convenient or necessary in the power series method. Shift the index so that the power under the summation sign is x^m. Check by writing the first few terms explicity.

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1}, \qquad \sum_{p=1}^{\infty} \frac{p^2}{(p+1)!} x^{p+4}$$

16–19 CAS PROBLEMS. IVPs

Solve the initial value problem by a power series. Graph the partial sums of the powers up to and including x^5. Find the value of the sum s (5 digits) at x_1.

16. $y' + 4y = 1, \quad y(0) = 1.25, \quad x_1 = 0.2$

17. $y'' + 3xy' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1,$
$x = 0.5$

18. $(1-x^2)y'' - 2xy' + 30y = 0, \quad y(0) = 0,$
$y'(0) = 1.875, \quad x_1 = 0.5$

19. $(x-2)y' = xy, \quad y(0) = 4, \quad x_1 = 2$

20. CAS Experiment. Information from Graphs of Partial Sums. In numerics we use partial sums of power series. To get a feel for the accuracy for various x, experiment with $\sin x$. Graph partial sums of the Maclaurin series of an increasing number of terms, describing qualitatively the "breakaway points" of these graphs from the graph of $\sin x$. Consider other Maclaurin series of your choice.

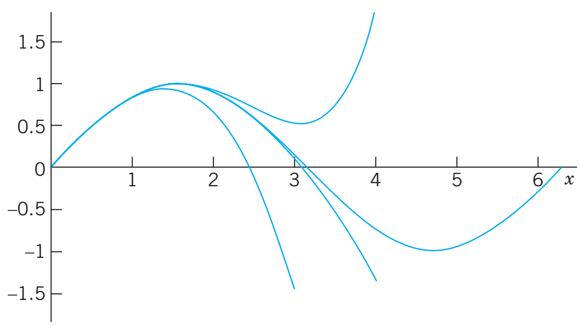

Fig. 106. CAS Experiment 20. $\sin x$ and partial sums s_3, s_5, s_7

5.2 Legendre's Equation. Legendre Polynomials $P_n(x)$

本題了解作法即可，不會考。

Legendre's differential equation[1]

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \qquad (n \text{ constant}) \tag{1}$$

is one of the most important ODEs in physics. It arises in numerous problems, particularly in boundary value problems for spheres (take a quick look at Example 1 in Sec. 12.10).

The equation involves a **parameter** n, whose value depends on the physical or engineering problem. So (1) is actually a whole family of ODEs. For $n = 1$ we solved it in Example 3 of Sec. 5.1 (look back at it). Any solution of (1) is called a **Legendre function**. The study of these and other "higher" functions not occurring in calculus is called the **theory of special functions**. Further special functions will occur in the next sections.

Dividing (1) by $1 - x^2$, we obtain the standard form needed in Theorem 1 of Sec. 5.1 and we see that the coefficients $-2x/(1 - x^2)$ and $n(n + 1)/(1 - x^2)$ of the new equation are analytic at $x = 0$, so that we may apply the power series method. Substituting

先假設y，再代入。

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{2}$$

and its derivatives into (1), and denoting the constant $n(n + 1)$ simply by k, we obtain

$$(1 - x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

It may help you to write out the first few terms of each series explicitly, as in Example 3 of Sec. 5.1; or you may continue as follows. To obtain the same general power x^s in all four series, set $m - 2 = s$ (thus $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives

黃字部分，令s=n-2，所以全部都是 x^s

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2} x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

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[1]ADRIEN-MARIE LEGENDRE (1752–1833), French mathematician, who became a professor in Paris in 1775 and made important contributions to special functions, elliptic integrals, number theory, and the calculus of variations. His book *Éléments de géométrie* (1794) became very famous and had 12 editions in less than 30 years.

Formulas on Legendre functions may be found in Refs. [GenRef1] and [GenRef10].

(Note that in the first series the summation begins with $s = 0$.) Since this equation with the right side 0 must be an identity in x if (2) is to be a solution of (1), the sum of the coefficients of each power of x on the left must be zero. Now $\boxed{x^0}$ occurs in the first and fourth series only, and gives [remember that $k = n(n+1)$]

$$\boxed{2 \cdot 1 a_2 + n(n+1)a_0 = 0.} \tag{3a}$$

$\boxed{x^1}$ occurs in the first, third, and fourth series and gives

$$\boxed{3 \cdot 2a_3 + [-2 + n(n+1)]a_1 = 0.} \tag{3b}$$

The higher powers $\boxed{x^2, x^3,} \cdots$ occur in all four series and give

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0. \tag{3c}$$

The expression in the brackets $[\cdots]$ can be written $(n-s)(n+s+1)$, as you may readily verify. Solving (3a) for a_2 and (3b) for a_3 as well as (3c) for a_{s+2}, we obtain the general formula

x^2, x^3以上

$$\boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\, a_s} \qquad (s = 0, 1, \cdots). \tag{4}$$

This is called a **recurrence relation** or **recursion formula**. (Its derivation you may verify with your CAS.) It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively

$$\boxed{\begin{aligned} a_2 &= -\frac{n(n+1)}{2!}a_0 & \qquad a_3 &= -\frac{(n-1)(n+2)}{3!}a_1 \\ a_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3}a_2 & a_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4}a_3 \\ &= \frac{(n-2)n(n+1)(n+3)}{4!}a_0 & &= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1 \end{aligned}}$$

and so on. By inserting these expressions for the coefficients into (2) we obtain

這是答案。

$$\boxed{y(x) = a_0 y_1(x) + a_1 y_2(x)} \tag{5}$$

where

$$\boxed{y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots} \tag{6}$$

$$\boxed{y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots.} \tag{7}$$

以下不必看。僅最後瞭解一下。

若n為奇數，y1是無限多項，y2是有限項。
若n為偶數，y1是有限項，y2是無限多項。

These series converge for $|x| < 1$ (see Prob. 4; or they may terminate, see below). Since (6) contains even powers of x only, while (7) contains odd powers of x only, the ratio y_1/y_2 is not a constant, so that y_1 and y_2 are not proportional and are thus linearly independent solutions. Hence (5) is a general solution of (1) on the interval $-1 < x < 1$.

Note that $x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardized ODE are no longer analytic. So it should not surprise you that we do not get a longer convergence interval of (6) and (7), unless these series terminate after finitely many powers. In that case, the series become polynomials.

Polynomial Solutions. Legendre Polynomials $P_n(x)$

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions. For special functions arising as solutions of ODEs this happens quite frequently, leading to various important families of polynomials; see Refs. [GenRef1], [GenRef10] in App. 1. For Legendre's equation this happens when the parameter n is a nonnegative integer because then the right side of (4) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0, \cdots$. Hence if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of such constants is done as follows. We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad (n \text{ a positive integer}) \tag{8}$$

(and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from (4), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leqq n-2). \tag{9}$$

The choice (8) makes $p_n(1) = 1$ for every n (see Fig. 107); this motivates (8). From (9) with $s = n - 2$ and (8) we obtain

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2}$$

Using $(2n)! = 2n(2n-1)(2n-2)!$ in the numerator and $n! = n(n-1)!$ and $n! = n(n-1)(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!\, n(n-1)(n-2)!}.$$

$n(n-1)2n(2n-1)$ cancels, so that we get

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!\, (n-2)!}.$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!}$$

and so on, and in general, when $n - 2m \geqq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}. \tag{10}$$

The resulting solution of Legendre's differential equation (1) is called the **Legendre polynomial** *of degree n* and is denoted by $P_n(x)$.

From (10) we obtain

$$\begin{aligned} P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!\,(n-1)!\,(n-2)!} x^{n-2} + - \cdots \end{aligned} \tag{11}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are (Fig. 107)

有限項時，調整係數後，y1或y2如右。

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x \\ &P_2(x) = \tfrac{1}{2}(3x^2 - 1), && P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \\ &P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned} \tag{11'}$$

下次小考不考本章，考p.343的 quadratic form

and so on. You may now program (11) on your CAS and calculate $P_n(x)$ as needed.

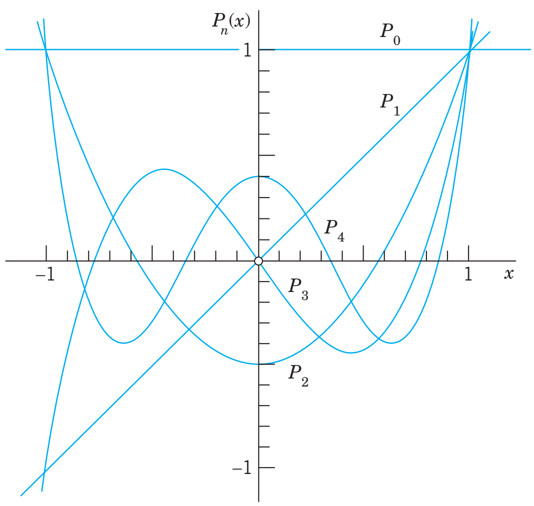

Fig. 107. Legendre polynomials

The Legendre polynomials $P_n(x)$ are **orthogonal** on the interval $-1 \leqq x \leqq 1$, a basic property to be defined and used in making up "Fourier–Legendre series" in the chapter on Fourier series (see Secs. 11.5–11.6).

PROBLEM SET 5.2

1–5 LEGENDRE POLYNOMIALS AND FUNCTIONS

1. Legendre functions for $n = 0$. Show that (6) with $n = 0$ gives $P_0(x) = 1$ and (7) gives (use $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$)

$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Verify this by solving (1) with $n = 0$, setting $z = y'$ and separating variables.

2. Legendre functions for $n = 1$. Show that (7) with $n = 1$ gives $y_2(x) = P_1(x) = x$ and (6) gives

$$y_1 = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$$
$$= 1 - \frac{1}{2}x\ln\frac{1+x}{1-x}.$$

3. Special n. Derive (11′) from (11).

4. Legendre's ODE. Verify that the polynomials in (11′) satisfy (1).

5. Obtain P_6 and P_7.

6–9 CAS PROBLEMS

6. Graph $P_2(x), \cdots, P_{10}(x)$ on common axes. For what x (approximately) and $n = 2, \cdots, 10$ is $|P_n(x)| < \frac{1}{2}$?

7. From what n on will your CAS no longer produce faithful graphs of $P_n(x)$? Why?

8. Graph $Q_0(x), Q_1(x)$, and some further Legendre functions.

9. Substitute $a_s x^s + a_{s+1}x^{s+1} + a_{s+2}x^{s+2}$ into Legendre's equation and obtain the coefficient recursion (4).

10. TEAM PROJECT. Generating Functions. Generating functions play a significant role in modern applied mathematics (see [GenRef5]). The idea is simple. If we want to study a certain sequence $(f_n(x))$ and can find a function

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x)u^n,$$

we may obtain properties of $(f_n(x))$ from those of G, which "generates" this sequence and is called a **generating function** of the sequence.

(a) Legendre polynomials. Show that

$$G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n \tag{12}$$

is a generating function of the Legendre polynomials. *Hint:* Start from the binomial expansion of $1/\sqrt{1 - v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

(b) Potential theory. Let A_1 and A_2 be two points in space (Fig. 108, $r_2 > 0$). Using (12), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}}$$
$$= \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m.$$

This formula has applications in potential theory. (Q/r is the electrostatic potential at A_2 due to a charge Q located at A_1. And the series expresses $1/r$ in terms of the distances of A_1 and A_2 from any origin O and the angle θ between the segments OA_1 and OA_2.)

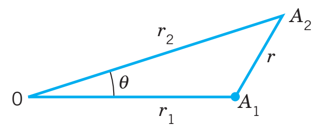

Fig. 108. Team Project 10

(c) Further applications of (12). Show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and $P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n - 1)/[2 \cdot 4 \cdots (2n)]$.

11–15 FURTHER FORMULAS

11. ODE. Find a solution of $(a^2 - x^2)y'' - 2xy' + n(n + 1)y = 0$, $a \neq 0$, by reduction to the Legendre equation.

12. Rodrigues's formula (13)[2] Applying the binomial theorem to $(x^2 - 1)^n$, differentiating it n times term by term, and comparing the result with (11), show that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]. \tag{13}$$

[2] OLINDE RODRIGUES (1794–1851), French mathematician and economist.