**13. Rodrigues's formula.** Obtain (11′) from (13).

**14. Bonnet's recursion.**[3] Differentiating (13) with respect to $u$, using (13) in the resulting formula, and comparing coefficients of $u^n$, obtain the *Bonnet recursion.*

(14) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - np_{n-1}(x),$

where $n = 1, 2, \cdots$. This formula is useful for computations, the loss of significant digits being small (except near zeros). Try (14) out for a few computations of your own choice.

**15. Associated Legendre functions $P_n^k(x)$** are needed, e.g., in quantum physics. They are defined by

**(15)** $$P_n^k(x) = (1 - x^2)^{k/2} \frac{d^k p_n(x)}{dx^k}$$

and are solutions of the ODE

**(16)** $$(1 - x^2)y'' - 2xy' + q(x)y = 0$$

where $q(x) = n(n+1) - k^2/(1 - x^2)$. Find $P_1^1(x)$, $P_2^1(x)$, $P_2^2(x)$, and $P_4^2(x)$ and verify that they satisfy (16).

# 5.3 Extended Power Series Method: Frobenius Method

Several second-order ODEs of considerable practical importance—the famous Bessel equation among them—have coefficients that are not analytic (definition in Sec. 5.1), but are "not too bad," so that these ODEs can still be solved by series (power series times a logarithm or times a fractional power of $x$, etc.). Indeed, the following theorem permits an extension of the power series method. The new method is called the **Frobenius method.**[4] Both methods, that is, the power series method and the Frobenius method, have gained in significance due to the use of software in actual calculations.

**THEOREM 1**

(1) 形式之方程式，可假設如(2)式來求解。

**Frobenius Method**

*Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE*

(1) $$\boxed{y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0}$$

*has at least one solution that can be represented in the form*

(2) $$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)} \qquad (a_0 \neq 0)$$

*where the exponent $r$ may be any (real or complex) number (and $r$ is chosen so that $a_0 \neq 0$).*

*The ODE* (1) *also has a second solution* (*such that these two solutions are linearly independent*) *that may be similar to* (2) (*with a different $r$ and different coefficients*) *or may contain a logarithmic term.* (Details in Theorem 2 below.)

[3] OSSIAN BONNET (1819–1892), French mathematician, whose main work was in differential geometry.

[4] GEORG FROBENIUS (1849–1917), German mathematician, professor at ETH Zurich and University of Berlin, student of Karl Weierstrass (see footnote, Sect. 15.5). He is also known for his work on matrices and in group theory.

In this theorem we may replace $x$ by $x - x_0$ with any number $x_0$. The condition $a_0 \neq 0$ is no restriction; it simply means that we factor out the highest possible power of $x$.

The singular point of (1) at $x = 0$ is often called a **regular singular point**, a term confusing to the student, which we shall not use.

For example, Bessel's equation (to be discussed in the next section)

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - \nu^2}{x^2}\right)y = 0 \qquad (\nu \text{ a parameter})$$

is of the form (1) with $b(x) = 1$ and $c(x) = x^2 - \nu^2$ analytic at $x = 0$, so that the theorem applies. This ODE could not be handled in full generality by the power series method.

Similarly, the so-called hypergeometric differential equation (see Problem Set 5.3) also requires the Frobenius method.

The point is that in (2) we have a power series times a single power of $x$ whose exponent $r$ is not restricted to be a nonnegative integer. (The latter restriction would make the whole expression a power series, by definition; see Sec. 5.1.)

The proof of the theorem requires advanced methods of complex analysis and can be found in Ref. [A11] listed in App. 1.

**Regular and Singular Points.** The following terms are practical and commonly used. A **regular point** of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

is a point $x_0$ at which the coefficients $p$ and $q$ are analytic. Similarly, a **regular point** of the ODE

$$\tilde{h}(x)y'' + \tilde{p}(x)y'(x) + \tilde{q}(x)y = 0$$

is an $x_0$ at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic and $\tilde{h}(x_0) \neq 0$ (so what we can divide by $\tilde{h}$ and get the previous standard form). Then the power series method can be applied. If $x_0$ is not a regular point, it is called a **singular point**.

## Indicial Equation, Indicating the Form of Solutions

We shall now explain the Frobenius method for solving (1). Multiplication of (1) by $x^2$ gives the more convenient form

$$\boxed{x^2y'' + xb(x)y' + c(x)y = 0.} \tag{1'}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$\boxed{b(x) = b_0 + b_1x + b_2x^2 + \cdots}, \qquad \boxed{c(x) = c_0 + c_1x + c_2x^2 + \cdots}$$

or we do nothing if $b(x)$ and $c(x)$ are polynomials. Then we differentiate (2) term by term, finding

$$\boxed{y'(x) = \sum_{m=0}^{\infty} (m + r)a_mx^{m+r-1} = x^{r-1}[ra_0 + (r + 1)a_1x + \cdots]}$$

$$\boxed{\begin{aligned} y''(x) &= \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_mx^{m+r-2} \\ &= x^{r-2}[r(r - 1)a_0 + (r + 1)ra_1x + \cdots]. \end{aligned}} \tag{2*}$$

y, y', y''代入，求最低次項x^r

By inserting all these series into (1′) we obtain

$$x^r[r(r-1)a_0 + \cdots] + (b_0 + b_1 x + \cdots)x^r(ra_0 + \cdots) + (c_0 + c_1 x + \cdots)x^r(a_0 + a_1 x + \cdots) = 0. \quad (3)$$

We now equate the sum of the coefficients of each power $x^r, x^{r+1}, x^{r+2}, \cdots$ to zero. This yields a system of equations involving the unknown coefficients $a_m$. The smallest power is $x^r$ and the corresponding equation is

$$[r(r-1) + b_0 r + c_0]a_0 = 0.$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives

$$\boxed{r(r-1) + b_0 r + c_0 = 0.} \quad \textbf{(4)}$$

This important quadratic equation is called the **indicial equation** of the ODE (1). Its role is as follows.

The Frobenius method yields a basis of solutions. One of the two solutions will always be of the form (2), where $r$ is a root of (4). The other solution will be of a form indicated by the indicial equation. There are three cases:

有3種case，第一種就直接做。第二、三種如例題。

***Case 1.*** Distinct roots not differing by an integer 1, 2, 3, ⋯.

***Case 2.*** A double root.

***Case 3.*** Roots differing by an integer 1, 2, 3, ⋯.

Cases 1 and 2 are not unexpected because of the Euler–Cauchy equation (Sec. 2.5), the simplest ODE of the form (1). Case 1 includes complex conjugate roots $r_1$ and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i \operatorname{Im} r_1$ is imaginary, so it cannot be a *real* integer. The form of a basis will be given in Theorem 2 (which is proved in App. 4), without a general theory of convergence, but convergence of the occurring series can be tested in each individual case as usual. Note that in Case 2 *we **must*** have a logarithm, whereas in Case 3 we *may* or *may not*.

**THEOREM 2**

**Frobenius Method. Basis of Solutions. Three Cases**

*Suppose that the ODE* (1) *satisfies the assumptions in Theorem* 1. *Let $r_1$ and $r_2$ be the roots of the indicial equation* (4). *Then we have the following three cases.*

***Case* 1. *Distinct Roots Not Differing by an Integer*.** *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad (5)$$

*and*

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \quad (6)$$

*with coefficients obtained successively from* (3) *with $r = r_1$ and $r = r_2$, respectively.*

case 2與3，解出y1後，通常設y2=u y1來求解。
如：例2、3

***Case 2. Double Root $r_1 = r_2 = r$.*** *A basis is*

$$y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{7}$$

*(of the same general form as before) and*

$$y_2(x) = y_1(x) \ln x + x^r(A_1x + A_2x^2 + \cdots) \qquad (x > 0). \tag{8}$$

***Case 3. Roots Differing by an Integer.*** *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \tag{9}$$

*(of the same general form as before) and*

$$y_2(x) = ky_1(x) \ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots), \tag{10}$$

*where the roots are so denoted that $r_1 - r_2 > 0$ and $k$ may turn out to be zero.*

## Typical Applications

Technically, the Frobenius method is similar to the power series method, once the roots of the indicial equation have been determined. However, (5)–(10) merely indicate the general form of a basis, and a second solution can often be obtained more rapidly by reduction of order (Sec. 2.1).

**EXAMPLE 1 Euler–Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm**

For the Euler–Cauchy equation (Sec. 2.5)

$$x^2y'' + b_0xy' + c_0y = 0 \qquad (b_0, c_0 \text{ constant})$$

substitution of $y = x^r$ gives the auxiliary equation

$$r(r - 1) + b_0r + c_0 = 0,$$

which is the indicial equation [and $y = x^r$ is a very special form of (2)!]. For different roots $r_1, r_2$ we get a basis $y_1 = x^{r_1}$, $y_2 = x^{r_2}$, and for a double root $r$ we get a basis $x^r$, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role.

**EXAMPLE 2 Illustration of Case 2 (Double Root)**

本題型式雖與 (1')有點不同，但仍可用Frobenius method求解

Solve the ODE

$$\boxed{x(x - 1)y'' + (3x - 1)y' + y = 0.} \tag{11}$$

(This is a special hypergeometric equation, as we shall see in the problem set.)

***Solution.*** Writing (11) in the standard form (1), we see that it satisfies the assumptions in Theorem 1. [What are $b(x)$ and $c(x)$ in (11)?] By inserting (2) and its derivatives (2*) into (11) we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty} (m + r)(m + r - 1)a_mx^{m+r} - \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_mx^{m+r-1}\\ &+ 3\sum_{m=0}^{\infty} (m + r)a_mx^{m+r} - \sum_{m=0}^{\infty} (m + r)a_mx^{m+r-1} + \sum_{m=0}^{\infty} a_mx^{m+r} = 0.\end{aligned} \tag{12}$$

The smallest power is $x^{r-1}$, occurring in the second and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1)-r]a_0 = 0, \qquad \text{thus} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.

把(12)之各項都化成x^s，然後來求解。有點複雜，需自行練習。

***First Solution.*** We insert this value $r = 0$ into (12) and equate the sum of the coefficients of the power $x^s$ to zero, obtaining

$$s(s-1)a_s - (s+1)sa_{s+1} + 3sa_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

***Second Solution.*** We get a second independent solution $y_2$ by the method of reduction of order (Sec. 2.1), substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9), Sec. 2.1, which we shall use in this example, instead of starting reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of $y'$ in (11) ***in standard form***. By partial fractions,

$$-\int p\,dx = -\int \frac{3x-1}{x(x-1)}\,dx = -\int\left(\frac{2}{x-1} + \frac{1}{x}\right)dx = -2\ln(x-1) - \ln x.$$

Hence (9), Sec. 2.1, becomes

$$u' = U = y_1^{-2}e^{-\int p\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = uy_1 = \frac{\ln x}{1-x}.$$

$y_1$ and $y_2$ are shown in Fig. 109. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$). ■

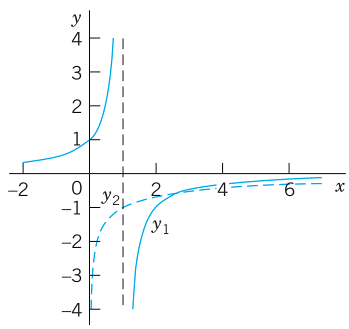


**Fig. 109.** Solutions in Example 2

## EXAMPLE 3 Case 3, Second Solution with Logarithmic Term

Solve the ODE

$$(13) \qquad (x^2 - x)y'' - xy' + y = 0.$$

***Solution.*** Substituting (2) and (2*) into (13), we have

$$(x^2 - x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} - x\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We now take $x^2$, $x$, and $x$ inside the summations and collect all terms with power $x^{m+r}$ and simplify algebraically,

$$\sum_{m=0}^{\infty} (m + r - 1)^2 a_m x^{m+r} - \sum_{m=0}^{\infty} (m + r)(m + r - 1) a_m x^{m+r-1} = 0.$$

In the first series we set $m = s$ and in the second $m = s + 1$, thus $s = m - 1$. Then

$$\sum_{s=0}^{\infty} (s + r - 1)^2 a_s x^{s+r} - \sum_{s=-1}^{\infty} (s + r + 1)(s + r) a_{s+1} x^{s+r} = 0. \tag{14}$$

The lowest power is $x^{r-1}$ (take $s = -1$ in the second series) and gives the indicial equation

$$r(r - 1) = 0.$$

先取r比較大的值。也是把各項都化成X^S

The roots are $\boxed{r_1 = 1 \text{ and } r_2 = 0}$. They differ by an integer. This is Case 3.

***First Solution.*** From (14) with $r = r_1 = 1$ we have

$$\boxed{\sum_{s=0}^{\infty} [s^2 a_s - (s + 2)(s + 1) a_{s+1}] x^{s+1} = 0.}$$

This gives the recurrence relation

$$\boxed{a_{s+1} = \frac{s^2}{(s + 2)(s + 1)} a_s} \qquad (s = 0, 1, \cdots).$$

Hence $\boxed{a_1 = 0, a_2 = 0,} \cdots$ successively. Taking $\boxed{a_0 = 1}$ we get as a first solution $\boxed{y_1 = x^{r_1} a_0 = x.}$

***Second Solution.*** Applying reduction of order (Sec. 2.1), we substitute $\boxed{y_2 = y_1 u = xu, y_2' = xu' + u}$ and $\boxed{y_2'' = xu'' + 2u'}$ into the ODE, obtaining

$$(x^2 - x)(xu'' + 2u') - x(xu' + u) + xu = 0.$$

$xu$ drops out. Division by $x$ and simplification give

$$(x^2 - x)u'' + (x - 2)u' = 0.$$

From this, using partial fractions and integrating (taking the integration constant zero), we get

$$\frac{u''}{u'} = -\frac{x - 2}{x^2 - x} = -\frac{2}{x} + \frac{1}{1 - x}, \qquad \ln u' = \ln\left|\frac{x - 1}{x^2}\right|.$$

Taking exponents and integrating (again taking the integration constant zero), we obtain

$$u' = \frac{x - 1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \ln x + \frac{1}{x}, \qquad \boxed{y_2 = xu = x \ln x + 1.}$$

$y_1$ and $y_2$ are linearly independent, and $y_2$ has a logarithmic term. Hence $y_1$ and $y_2$ constitute a basis of solutions for positive $x$. ■

本學期上課到此，考到這裡。

The Frobenius method solves the **hypergeometric equation**, whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

## PROBLEM SET 5.3

1. **WRITING PROJECT. Power Series Method and Frobenius Method.** Write a report of 2–3 pages explaining the difference between the two methods. No proofs. Give simple examples of your own.

### 2–13 FROBENIUS METHOD

Find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions. Show the details of your work.

2. $(x+2)^2 y'' + (x+2)y' - y = 0$
3. $xy'' + 2y' + xy = 0$
4. $xy'' + y = 0$
5. $xy'' + (2x+1)y' + (x+1)y = 0$
6. $xy'' + 2x^3 y' + (x^2-2)y = 0$
7. $y'' + (x-1)y = 0$
8. $xy'' + y' - xy = 0$
9. $2x(x-1)y'' - (x+1)y' + y = 0$
10. $xy'' + 2y' + 4xy = 0$
11. $xy'' + (2-2x)y' + (x-2)y = 0$
12. $x^2y'' + 6xy' + (4x^2+6)y = 0$
13. $xy'' + (1-2x)y' + (x-1)y = 0$
14. **TEAM PROJECT. Hypergeometric Equation, Series, and Function. Gauss's hypergeometric ODE**[5] is

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{15}$$

Here, $a$, $b$, $c$ are constants. This ODE is of the form $p_2 y'' + p_1 y' + p_0 y = 0$, where $p_2$, $p_1$, $p_0$ are polynomials of degree 2, 1, 0, respectively. These polynomials are written so that the series solution takes a most practical form, namely,

$$\begin{aligned} y_1(x) = 1 &+ \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 \\ &+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots. \end{aligned} \tag{16}$$

This series is called the **hypergeometric series**. Its sum $y_1(x)$ is called the **hypergeometric function** and is denoted by $F(a, b, c; x)$. Here, $c \neq 0, -1, -2, \cdots$. By choosing specific values of $a$, $b$, $c$ we can obtain an incredibly large number of special functions as solutions of (15) [see the small sample of elementary functions in part (c)]. This accounts for the importance of (15).

**(a) Hypergeometric series and function.** Show that the indicial equation of (15) has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method gives (16). Motivate the name for (16) by showing that

$$F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x) = \frac{1}{1-x}.$$

**(b) Convergence.** For what $a$ or $b$ will (16) reduce to a polynomial? Show that for any other $a$, $b$, $c$ $(c \neq 0, -1, -2, \cdots)$ the series (16) converges when $|x| < 1$.

**(c) Special cases.** Show that

$$\begin{aligned} (1+x)^n &= F(-n, b, b; -x), \\ (1-x)^n &= 1 - nxF(1-n, 1, 2; x), \\ \arctan x &= xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2) \\ \arcsin x &= xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2), \\ \ln(1+x) &= xF(1, 1, 2; -x), \\ \ln\frac{1+x}{1-x} &= 2xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2). \end{aligned}$$

Find more such relations from the literature on special functions, for instance, from [GenRef1] in App. 1.

**(d) Second solution.** Show that for $r_2 = 1 - c$ the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$\begin{aligned} y_2(x) = x^{1-c}\Bigg(1 &+ \frac{(a-c+1)(b-c+1)}{1!\,(-c+2)}x \\ &+ \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!\,(-c+2)(-c+3)}x^2 \\ &+ \cdots\Bigg). \end{aligned} \tag{17}$$

Show that

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

**(e) On the generality of the hypergeometric equation.** Show that

$$(t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0 \tag{18}$$

[5] CARL FRIEDRICH GAUSS (1777–1855), great German mathematician. He already made the first of his great discoveries as a student at Helmstedt and Göttingen. In 1807 he became a professor and director of the Observatory at Göttingen. His work was of basic importance in algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, numeric analysis, astronomy, geodesy, electromagnetism, and theoretical mechanics. He also paved the way for a general and systematic use of complex numbers.

with $\dot{y} = dy/dt$, etc., constant $A, B, C, D, K$, and $t^2 + At + B = (t - t_1)(t - t_2)$, $t_1 \neq t_2$, can be reduced to the hypergeometric equation with independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

and parameters related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$. From this you see that (15) is a "normalized form" of the more general (18) and that various cases of (18) can thus be solved in terms of hypergeometric functions.

### 15–20 HYPERGEOMETRIC ODE

Find a general solution in terms of hypergeometric functions.

**15.** $2x(1 - x)y'' - (1 + 6x)y' - 2y = 0$

**16.** $x(1 - x)y'' + (\frac{1}{2} + 2x)y' - 2y = 0$

**17.** $4x(1 - x)y'' + y' + 8y = 0$

**18.** $4(t^2 - 3t + 2)\ddot{y} - 2\dot{y} + y = 0$

**19.** $2(t^2 - 5t + 6)\ddot{y} + (2t - 3)\dot{y} - 8y = 0$

**20.** $3t(1 + t)\ddot{y} + t\dot{y} - y = 0$

# 5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

One of the most important ODEs in applied mathematics in **Bessel's equation**,[6]

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1}$$

where the parameter $\nu$ (nu) is a given real number which is positive or zero. Bessel's equation often appears if a problem shows cylindrical symmetry, for example, as the membranes in Sec.12.9. The equation satisfies the assumptions of Theorem 1. To see this, divide (1) by $x^2$ to get the standard form $y'' + y'/x + (1 - \nu^2/x^2)y = 0$. Hence, according to the Frobenius theory, it has a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0). \tag{2}$$

Substituting (2) and its first and second derivatives into Bessel's equation, we obtain

$$\sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m + r)a_m x^{m+r}$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of $x^{s+r}$ to zero. Note that this power $x^{s+r}$ corresponds to $m = s$ in the first, second, and fourth series, and to $m = s - 2$ in the third series. Hence for $s = 0$ and $s = 1$, the third series does not contribute since $m \geqq 0$.

[6] FRIEDRICH WILHELM BESSEL (1784–1846), German astronomer and mathematician, studied astronomy on his own in his spare time as an apprentice of a trade company and finally became director of the new Königsberg Observatory.

Formulas on Bessel functions are contained in Ref. [GenRef10] and the standard treatise [A13].